# SOME INTEGRALGEOMETRIC THEOREMS(1)

BY

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1. **Introduction.** Assuming that $A$ and $B$ are analytic subsets of Euclidean $n$-space $E_n$, we consider for each isometry $f$ of $E_n$ the intersection $A \cap f(B)$. We prove (Theorem 5.7) that certain rectifiability properties of $A$ in dimension $k$ and of $B$ in dimension $l$, where $k+l \geqq n$, imply corresponding rectifiability properties of $A \cap f(B)$ in dimension $k+l-n$ for almost all $f$, and (Theorem 6.2) that in the presence of such rectifiability properties the integral over the group of isometries of the $k+l-n$ dimensional Hausdorff measure of $A \cap f(B)$ is proportional to the product of the $k$ dimensional Hausdorff measure of $A$ and the $l$ dimensional Hausdorff measure of $B$.

2. **Definitions.** The notational conventions used in this paper are in general consistent with those of [F2], [F3], and [F4].

2.1 Definition. If $n$ is a positive integer, then $E_n$ is the Euclidean space of dimension $n$, $\mathcal{L}_n$ is the $n$ dimensional Lebesgue measure over $E_n$, $G_n$ is the group of all orthogonal transformations of $E_n$, and $\phi_n$ is the Haar measure over $G_n$ such that $\phi_n(G_n)=1$.

If $z \in E_n$, then $T_z$ is the translation of $E_n$ defined by the formula

$$T_z(x) = z + x \text{ for } x \in E_n.$$

If $a \in E_n$ and $r > 0$, then

$$K(a, r) = E_n \cap \{x \mid |x-a| < r\}$$

is the open sphere with center $a$ and radius $r$.

2.2 Definition. If $\mu$ is a measure over $X$ and $\nu$ is a measure over $Y$, then the measure $\mu \otimes \nu$ over the cartesian product $X \times Y$ is defined as follows: For $P \subset X \times Y$, $(\mu \otimes \nu)(P)$ is the infimum of all numbers of the form

$$\sum_{i=1}^{\infty} \mu(A_i) \cdot \nu(B_i),$$

where $A_1, A_2, A_3, \cdots$ are $\mu$ measurable subsets of $X$; $B_1, B_2, B_3, \cdots$ are $\nu$ measurable subsets of $Y$; and

$$P \subset \bigcup_{i=1}^{\infty} A_i \times B_i.$$

[It is agreed here, as is usual in measure theory, that $0 \cdot \infty = \infty \cdot 0 = 0$.]

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2.3 REMARK. There is a natural one-to-one correspondence between the group of isometries of $E_n$ and the cartesian product space $E_n \times G_n$; in fact each isometry is uniquely representable in the form

$$T_z \circ R,$$

with $z \in E_n$ and $R \in G_n$. This correspondence associates with the measure $L_n \otimes \phi_n$ over $E_n \times G_n$ a certain measure over the group of isometries, which is readily seen to be a two-sided Haar measure. The integral of a function $g$ with respect to this Haar measure equals

$$\int g(T_z \circ R) d(L_n \otimes \phi_n)(z, R).$$

2.4 DEFINITION. If $k < n$ are positive integers, then the functions

$$\overset{k}{p}_n: E_n \to E_k \quad \text{and} \quad \overset{k}{\eta}_n: E_k \to E_n$$

are defined by the formulae

$$\overset{k}{p}_n(x) = (x_1, \cdots, x_k) \in E_k \qquad \text{for } x = (x_1, \cdots, x_n) \in E_n,$$
$$\overset{k}{\eta}_n(y) = (y_1, \cdots, y_k, 0, \cdots, 0) \in E_n \qquad \text{for } y = (y_1, \cdots, y_k) \in E_k.$$

2.5 DEFINITION. If $m < n$ are positive integers, then $\Lambda_n^m$ is the set of all $m$ dimensional planes contained in $E_n$,

$$\overset{m}{Z}_n = E_n \cap \{x \mid x_i = 0 \text{ for } i = 1, \cdots, n - m\} \in \overset{m}{\Lambda}_n,$$

and the function

$$\overset{m}{\lambda}_n : G_n \times E_{n-m} \to \overset{m}{\Lambda}_n$$

is defined by the formula

$$\overset{m}{\lambda}_n(R, z) = (R \circ T_{\overset{n-m}{\eta}_n(z)})(\overset{m}{Z}_n) \quad \text{for } (R, z) \in G_n \times E_{n-m}.$$

2.6 REMARK. The function $\lambda_n^m$ maps $G_n \times E_{n-m}$ onto $\Lambda_n^m$. Associating with each subset of $\Lambda_n^m$ the $\phi_n \otimes \mathcal{L}_{n-m}$ measure of its counterimage under $\lambda_n^m$, we obtain a measure over $\Lambda_n^m$ which is readily seen to be invariant under the transformations of $\Lambda_n^m$ induced by the isometries of $E_n$. The integral of a function $g$ with respect to this Haar measure equals

$$\int (g \circ \overset{m}{\lambda}_n) d(\phi_n \otimes \mathcal{L}_{n-m}).$$

2.7 REMARK. If $R \in G_n$ and $z \in E_{n-m}$, then

$$\overset{m}{\lambda}_n(R, z) = E_n \cap \{x \mid (\overset{n-m}{p}_n \circ \operatorname{inv} R)(x) = z\}.$$

2.8 Definition. In terms of the function $\Gamma$ of classical analysis we define the functions $\alpha$, $\beta$, $\gamma$ by the formulae

$$\alpha(k) = \frac{2^k}{\Gamma(k+1)} \Gamma\left(\frac{1}{2}\right)^{k-1} \Gamma\left(\frac{k+1}{2}\right) \qquad \text{for } k \geqq 0,$$

$$\beta(n, k) = \frac{\alpha(k)\alpha(n-k)}{\alpha(n)\binom{n}{k}} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} \qquad \text{for } n \geqq k \geqq 0,$$

$$\gamma(n, k, l) = \frac{\beta(n, k)\beta(n, l)}{\beta(n, k+l-n)\beta(2n-k-l, n-l)} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{k+l-n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}$$

$$\text{for } n \geqq k \geqq 0,\ n \geqq l \geqq 0,\ k+l \geqq n.$$

2.9 Definition. If $X$ is a metric metrized by $\rho$, then

$$\operatorname{diam}_\rho (S) = \sup_{x \in S, y \in S} \rho(x, y)$$

for any non-empty set $S \subset X$; if $S$ is the empty set, then $\operatorname{diam}_\rho (S) = 0$. Furthermore if $k \geqq 0$ and $S \subset X$, then

$$\overset{k}{\chi}_\rho(S) = \alpha(k)2^{-k}[\operatorname{diam}_\rho (S)]^k.$$

If $A \subset X$, then the $k$ dimensional Hausdorff measure

$$\overset{k}{\mathcal{H}}_\rho(A)$$

is defined as follows: For each $r > 0$ consider the infimum of all numbers of the form

$$\sum_{S \in F} \overset{k}{\chi}_\rho(S),$$

where $F$ is a countable covering of $A$ with

$$\sup_{S \in F} \operatorname{diam}_\rho (S) \leqq r.$$

As $r$ decreases, this infimum does not decrease; its limit, as $r$ approaches 0, equals $\mathcal{H}_\rho^k(A)$.

We further agree that if $X = E_n$ and $\rho$ is the usual metric of $E_n$, then $\mathcal{H}_\rho^k$ shall be designated by

$$\overset{k}{\mathcal{H}}_n.$$

2.10 Remark. $\mathcal{H}_\rho^0(A)$ equals the number (possibly $\infty$) of elements of $A$.

2.11 Definition. A function $f$ on $E_k$ to $E_n$ is *Lipschitzian* if and only if there exists an $M<\infty$ such that

$$|f(x) - f(x')| \leqq M|x - x'| \text{ whenever } x \in E_k \text{ and } x' \in E_k.$$

2.12 Definition. A subset $A$ of $E_n$ is *$k$ rectifiable* if and only if there exists a Lipschitzian function on $E_k$ to $E_n$ which maps some bounded subset of $E_k$ onto $A$.

2.13 Definition. A subset $A$ of $E_n$ is *Hausdorff $k$ rectifiable* if and only if

$$\overset{k}{\mathcal{H}}_n(A) < \infty$$

and there exist $k$ rectifiable subsets $B_1, B_2, B_3, \cdots$ of $E_n$ such that

$$\overset{k}{\mathcal{H}}_n\left(A - \bigcup_{i=1}^{\infty} B_i\right) = 0.$$

2.14 Remark. A set is Hausdorff $k$ rectifiable if and only if it is $(\overset{k}{\mathcal{H}}_n, k)$ rectifiable, as defined in [F3].

When discussing $k$ rectifiable or Hausdorff $k$ rectifiable subsets of $E_n$ in this paper, we shall always make the tacit assumption that $k<n$.

2.15 Definition. For each non-empty set $U\subset E_n$ we let $d_U$ be the function on $E_n$ such that

$$d_U(x) = \inf_{y\in U} |y - x|.$$

2.16 Definition. Suppose $A\subset E_n$, $a\in E_n$, and $U\in\Lambda_n^k$. Then we say that

*$A$ is restricted at $a$ by $U$*

if and only if $a\in U$ and for each $\epsilon>0$ there is a $\delta>0$ such that

$$A \cap K(a, \delta) \subset \{x \mid d_U(x) \leqq \epsilon |x - a|\}.$$

2.17 Remark. The preceding definition is closely related to Definition 2.28 of [F3]. Using the notation of [F3], we note that if $R\in G_n$, $a\in G_n$ and $U$ is the $k$ dimensional plane through $a$ perpendicular to $\square_n^{n-k}(R, a)$, then

$$|\overset{n-k}{P_R}(x - a)| = d_U(x) \text{ for } x \in E_n,$$

and consequently

$$\overset{n-k}{\Diamond_n}(R, \eta, a) = \{x \mid d_U(x) > (1 + \eta^2)^{-1/2} |x - a|\}$$

whenever $0<\eta<\infty$. Taking account of this connection and applying Theorems 3.7, 4.7, 5.1, 5.2, 5.7 of [F3] together with the proposition that if $f$ is a Lipschitzian function on $E_k$ to $E_n$ then $\mathcal{L}_k$ almost all of the set of those points of $E_k$ at which $f$ has a nonsingular differential can be represented as a countable union of compact sets on each of which $f$ is univalent, one readily obtains the following two results:

(1) *If* $\mathcal{H}_n^k(A) < \infty$ *and if for* $\mathcal{H}_n^k$ *almost all a in A it is true that A is restricted at a by some k dimensional plane, then A is Hausdorff k rectifiable.*

(2) *If A is a Hausdorff k rectifiable subset of* $E_n$, *then there are compact k rectifiable subsets* $A_1, A_2, A_3 \cdots$ *of* $E_n$ *such that*

$$\mathcal{H}_n^k\left(A - \bigcup_{i=1}^{\infty} A_i\right) = 0$$

*and such that for each positive integer i and each point* $a \in A_i$ *it is true that* $A_i$ *is restricted at a by some k dimensional plane.*

2.18 Definition. If $\mu$ is a measure over $X$ and $g$ is a function on $X$ to the extended real number system, then

$$\int^* g d\mu$$

is the upper Lebesgue integral of $g$ with respect to $\mu$; it equals the infimum of

$$\int h d\mu,$$

where $h$ is a $\mu$ integrable function such that

$$g(x) \leqq h(x) \leqq \infty \quad \text{for } x \in X.$$

3. **Some inequalities involving Hausdorff measure.** The central result of this section is Theorem 3.2. Since this proposition appears to be of basic interest for the theory of Hausdorff measure, it is formulated and proved in somewhat greater generality than the applications in the present paper require. For the immediate purpose it would be sufficient to consider only the special case in which $Z = E_u$; then Lemma 3.1 could be replaced by the isodiametric inequality.

If the space $Y$ in Theorem 3.2 consists of a single point and $\psi(Y) = 1$, then $X \times Y$, $\mathcal{H}_\rho^{u+v} \otimes \psi$, and $\mathcal{H}_\tau^v \otimes \psi$ may obviously be replaced by $X$, $\mathcal{H}_\rho^{u+v}$, and $\mathcal{H}_\tau^v$ respectively; thus $f$ becomes a function on $X$ to $Z$ with Lipschitz constant $M$, $P$ becomes a subset of $X$, and there results the inequality

$$\int^* \mathcal{H}_\rho^v(\{x \mid x \in P \text{ and } f(x) = z\}) d\mathcal{H}_\tau^u z \leqq M^u \frac{\alpha(u)\alpha(v)}{\alpha(u+v)} \mathcal{H}_\rho^{u+v}(P).$$

Two special cases of this inequality have been established previously:

Eilenberg treated in [E] the case in which $u = 1$, $Z = E_1$ and, for all $x \in X$, $f(x)$ equals the distance between $x$ and a fixed subset of $X$.

Besicovitch and Moran considered in Theorem 1 of [BM] the case in which $X=E_2$, $Z=E_1$, $f=p_1^2$ and the subset $P$ of $E_2$ is the cartesian product of two subsets $A$ and $B$ of $E_1$ [when convenient we identify $E_m \times E_n$ and $E_{m+n}$] with $\mathcal{H}_1^u(A)<\infty$ and $\mathcal{H}_1^v(B)<\infty$; here the inequality reduces to

$$\mathcal{H}_1^u(A)\,\mathcal{H}_1^v(B) \leqq \frac{\alpha(u)\alpha(v)}{\alpha(u+v)}\,\mathcal{H}_2^{u+v}(A \times B).$$

3.1 Lemma. *Suppose $\tau$ metrizes $Z$, $A \subset Z$, $u \geqq 0$, and*

$$\mathcal{H}_\tau^u(A) < \infty.$$

*For $s>1$ and $r>0$, let $B(s, r)$ be the set of all those points $z$ of $Z$ such that*

$$\mathcal{H}_\tau^u(A \cap W) \leqq s\chi_\tau^u(W)$$

*whenever $z \in W \subset Z$ and* $\operatorname{diam}_\tau (W) < r$. *Then*:

(1) *For $s>1$ and $r>0$, $B(s, r)$ is a closed subset of $Z$.*

(2) $\mathcal{H}_\tau^u[A - \bigcap_{s>1}\bigcup_{r>0} B(s, r)] = 0.$

The first of these two statements is subject to routine verification, while the second can be proved very much like Theorem 3.5 of [F3]. We omit the details.

3.2 Theorem. *If*

(i) *$X$, $Y$, $Z$ are separable and complete metric spaces, metrized by $\rho$, $\sigma$, $\tau$ respectively,*

(ii) *$f$ is a Baire function on $X \times Y$ to $Z$, $0<M<\infty$, and*

$$\tau[f(x, y), f(x', y)] \leqq M\rho(x, x')$$

*whenever, $x \in X$, $x' \in X$, $y \in Y$,*

(iii) *$\psi$ is such a measure over $Y$ that all Borel subsets of $Y$ are $\psi$ measurable and every $\psi$ measurable set is contained in a Borel set of equal $\psi$ measure,*

(iv) *$u \geqq 0$, $v \geqq 0$, $P \subset X \times Y$ and $f(P)$ is the union of countably many sets of finite $\mathcal{H}_\tau^u$ measure.*

*Then*

$$\int_{Z\times Y}^{*} \mathcal{H}_\rho^v(\{x \mid (x, y) \in P \text{ and } f(x, y) = z\})\,d(\mathcal{H}_\tau^u \otimes \psi)(z, y)$$

$$\leqq M^u \frac{\alpha(u)\alpha(v)}{\alpha(u+v)}\,(\mathcal{H}_\rho^{u+v} \otimes \psi)(P).$$

**Proof.** We let

$$q = M^u \frac{\alpha(u)\alpha(v)}{\alpha(u+v)}$$

and suppose that $s>1$.

Since $f(P)$ is the union of countably many sets of finite $\mathcal{H}_\tau^u$ measure, Lemma 3.1 enables us to construct a sequence of disjoint Borel sets $Z_1, Z_2, Z_3, \cdots$ of finite $\mathcal{H}_\tau^u$ measure and a sequence of positive numbers $r_1, r_2, r_3 \cdots$ such that

$$\mathcal{H}_\tau^u\left[f(P) - \bigcup_{i=1}^{\infty} Z_i\right] = 0$$

and such that for each positive integer $i$ the inequality

$$\mathcal{H}_\tau^u(W) \leqq s\chi_\tau^u(W)$$

holds whenever $W \subset Z_i$ and $\operatorname{diam}_\tau (W) < r_i$.

Associating with each set $Q \subset X \times Y$ the function $g_Q$ on $Z \times Y$ by the formula

$$g_Q(z, y) = \mathcal{H}_\rho^v(\{x \mid (x, y) \in Q \text{ and } f(x, y) = z\})$$

for $(z, y) \in Z \times Y$, we divide the remainder of the argument into four parts.

*Part* 1. *If $i$ is a positive integer, $F$ is a countable family of closed subsets of $X$ such that*

$$\sup_{C \in F} \operatorname{diam}_\rho (C) < r_i/M,$$

*$h$ is the function on $Z \times Y$ such that*

$$h(z, y) = \sum_{C \in F, z \in f(C \times \{y\})} \chi_\rho^v(C)$$

*for $(z, y) \in Z \times Y$, and $B$ is a Borel subset of $Y$, then*

$$\int_{Z_i \times B} hd(\mathcal{H}_\tau^u \otimes \psi) \leqq sq \sum_{C \in F} \chi_\rho^{u+v}(C)\psi(B).$$

**Proof.** For each $C \in F$ we define

$$H_C = \{(z, y) \mid z \in f(C \times \{y\})\}$$

and let $h_C$ be the characteristic function of $H_C$. Noting that the natural projection of $X \times Y \times Z$ onto $Z \times Y$ maps the Borel set

$$\{(x, y, z) \mid x \in C \text{ and } f(x, y) = z\}$$

onto $H_C$, we find that $H_C$ is an analytic set and hence that $h_C$ is an $\mathcal{H}_\tau^u \otimes \psi$ measurable function.

Next we observe that

$$h = \sum_{C \in F} \chi_\rho^v(C) h_C.$$

Thus $h$ is a non-negative $\mathcal{H}_\tau^u \otimes \psi$ measurable function, and it will be sufficient to show that

$$\chi_\rho^v(C) \int_{Z_i \times B} h_C d(\mathcal{H}_\tau^u \otimes \psi) \leqq sq\chi_\rho^{u+v}(C)\psi(B)$$

whenever $C \in F$.

For this purpose we henceforth fix $C \in F$ and assume that $\psi(B) < \infty$. Inasmuch as $\mathcal{H}_\tau^u(Z_i) < \infty$, the Fubini Theorem implies that

$$\int_{Z_i \times B} h_C d(\mathcal{H}_\tau^u \otimes \psi) = \int_B \int_{Z_i} h_C(z, y) d\mathcal{H}_\tau^u z d\psi y$$
$$= \int_B \mathcal{H}_\tau^u [Z_i \cap f(C \times \{y\})] d\psi y.$$

Furthermore if $y \in B$ then

$$\operatorname{diam}_\tau [f(C \times \{y\})] \leqq M \operatorname{diam}_\rho (C) < r_i,$$

and consequently

$$\mathcal{H}_\tau^u [Z_i \cap f(C \times \{y\})] \leqq s\chi_\tau^u [f(C \times \{y\})] \leqq sM^u \chi_\rho^u(C).$$

We conclude that

$$\chi_\rho^v(C) \int_{Z_i \times B} h_C d(\mathcal{H}_\tau^u \otimes \psi) \leqq \chi_\rho^v(C) s M^u \chi_\rho^u(C)\psi(B)$$
$$= sq\chi_\rho^{u+v}(C)\psi(B).$$

*Part 2. If $i$ is a positive integer, $A$ is a Borel subset of $X$, and $B$ is a Borel subset of $Y$, then*

$$\int_{Z_i \times Y}^* g_{A \times B} d(\mathcal{H}_\tau^u \otimes \psi) \leqq sq\mathcal{H}_\rho^{u+v}(A)\psi(B).$$

**Proof.** We consider a sequence of countable closed coverings $F_1, F_2, F_3, \cdots$ of $A$ such that

$$\sup_{C \in F_j} \operatorname{diam}_\rho (C) < r_i/M \text{ for } j = 1, 2, 3, \cdots,$$
$$\lim_{j \to \infty} \sup_{C \in F_j} \operatorname{diam}_\rho (C) = 0,$$
$$\lim_{j \to \infty} \sum_{C \in F_j} \chi_\rho^{u+v}(C) = \mathcal{H}_\rho^{u+v}(A).$$

Associating with each covering $F_j$ the function $h_j$ on $Z \times Y$ such that

$$h_j(z, y) = \sum_{C \in F_j, z \in f(C \times \{y\})} \chi_\rho^v(C)$$

for $(z, y) \in Z \times Y$, we infer from Fatou's Lemma and Part 1 that

$$\int_{Z_i \times B} \liminf_{j\to\infty} h_j(z, y)\, d(\mathcal{H}_r^u \otimes \psi)(z, y)$$

$$\leqq \liminf_{j\to\infty} \int_{Z_i \times B} h_j\, d(\mathcal{H}_r^u \otimes \psi) \leqq sq\mathcal{H}_\rho^{u+v}(A)\psi(B).$$

Finally we note that if $(z, y) \in Z_i \times Y$ and

$$g_{A\times B}(z, y) \neq 0,$$

then $y \in B$ and

$$\{x \mid (x, y) \in A \times B \text{ and } f(x, y) = z\} = A \cap \{x \mid f(x, y) = z\}$$

$$\subset \bigcup_{C \in F_j} C \cap \{x \mid f(x, y) = z\} = \bigcup_{C \in F_j, z \in f(C \times \{y\})} C$$

for every positive integer $j$, hence

$$g_{A\times B}(z, y) \leqq \liminf_{j\to\infty} h_j(z, y).$$

*Part* 3. *If $i$ is a positive integer and $Q \subset X \times Y$, then*

$$\int^*_{Z_i \times Y} g_Q\, d(\mathcal{H}_r^u \otimes \psi) \leqq sq(\mathcal{H}_\rho^{u+v} \otimes \psi)(Q).$$

**Proof.** Let $\epsilon > 0$. We choose Borel subsets $A_1, A_2, A_3, \cdots$ of $X$ and Borel subsets $B_1, B_2, B_3, \cdots$ of $Y$ such that

$$Q \subset \bigcup_{j=1}^{\infty} A_j \times B_j$$

and

$$\sum_{j=1}^{\infty} \mathcal{H}_\rho^{u+v}(A_j)\psi(B_j) \leqq (\mathcal{H}_\rho^{u+v} \otimes \psi)(Q) + \epsilon.$$

Noting that

$$g_Q(z, y) \leqq \sum_{j=1}^{\infty} g_{A_j \times B_j}(z, y) \text{ for } (z, y) \in (Z \times Y),$$

we use Part 2 to infer that

$$\int^*_{Z_i \times Y} g_Q\, d(\mathcal{H}_r^u \otimes \psi) \leqq \sum_{j=1}^{\infty} \int^*_{Z_i \times Y} g_{A_j \times B_j}\, d(\mathcal{H}_r^u \otimes \psi)$$

$$\leqq \sum_{j=1}^{\infty} sq\ \mathcal{H}_\rho^{u+v}(A_j)\psi(B_j) \leqq sq\ [(\mathcal{H}_\rho^{u+v} \otimes \psi)(Q) + \epsilon].$$

*Part* 4. $\int^* g_P(\mathcal{H}^u_\tau \otimes \psi) \leqq sq(\mathcal{H}^{u+v}_\rho \otimes \psi)(P)$.

**Proof.** Letting

$$T = f(P) - \bigcup_{i=1}^{\infty} Z_i,$$

we have $\mathcal{H}^u_\tau(T) = 0$ and consequently

$$(\mathcal{H}^u_\tau \otimes \psi)(T \times Y) = 0.$$

Inasmuch as

$$\{(z, y) \mid g_P(z, y) \neq 0\} \subset [f(P) \times Y] \subset (T \times Y) \cup \bigcup_{i=1}^{\infty} (Z_i \times Y),$$

we infer that

$$\int^*_{Z \times Y} g_P d(\mathcal{H}^u_\tau \otimes \psi) \leqq \sum_{i=1}^{\infty} \int^*_{Z_i \times Y} g_P d(\mathcal{H}^u_\tau \otimes \psi).$$

Considering next, for each positive integer $i$, the set

$$R_i = \{(x, y) \mid f(x, y) \in Z_i\},$$

we see that

$$g_P(z, y) = g_{P \cap R_i}(z, y) \quad \text{whenever } (z, y) \in Z_i \times Y,$$

and use Part 3 to obtain

$$\int^*_{Z_i \times Y} g_P d(\mathcal{H}^u_\tau \otimes \psi) = \int^*_{Z_i \times Y} g_{P \cap R_i} d(\mathcal{H}^u_\tau \otimes \psi) \leqq sq\,(\mathcal{H}^{u+v}_\rho \otimes \psi)(P \cap R_i).$$

The statement of Part 4 now follows from the inequality

$$\sum_{i=1}^{n} (\mathcal{H}^{u+v}_\rho \otimes \psi)(P \cap R_i) \leqq (\mathcal{H}^{u+v}_\rho \otimes \psi)(P),$$

which is true because $R_1, R_2, R_3, \cdots$ are disjoint Borel subsets of $X \times Y$.

Since $s$ is an arbitrary number greater than 1, the theorem is an immediate consequence of Part 4.

3.3 Theorem. *If $u < n$ are positive integers, $k \geqq u$, and $P \subset E_n \times G_n$, then*

$$\int^*_{E_u \times G_n} \mathcal{H}^{k-u}_n [\{a \mid (a, R) \in P\} \cap \lambda^{n-u}_n(R, z)] d(\mathcal{L}_u \otimes \phi_n)(z, R)$$
$$\leqq \frac{\alpha(u)\alpha(k-u)}{\alpha(k)} (\mathcal{H}^k_n \otimes \phi_n)(P).$$

**Proof.** Letting $f$ be the function on $E_n \times G_n$ to $E_u$ such that

$$f(a, R) = (p_n^u \circ \operatorname{inv} R)(a)$$

for $(a, R) \in E_n \times G_n$, we see that $f$ is a continuous function and that

$$|f(a, R) - f(a', R)| \leqq |a - a'|$$

whenever $a \in E_n$, $a' \in E_n$, and $R \in G_n$. Accordingly the preceding theorem (applied with $v = k - u$) yields the inequality

$$\int^* \mathcal{H}_n^{k-u}(\{a \mid (a, R) \in P \text{ and } (p_n^u \circ \operatorname{inv} R)(a) = z\}) d(\mathcal{L}_u \otimes \phi_n)(z, R)$$

$$\leqq \frac{\alpha(u)\alpha(k-u)}{\alpha(k)} (\mathcal{H}_n^k \otimes \phi_n)(P),$$

and reference to 2.7 completes the proof.

3.4 COROLLARY. *If $u < n$ are positive integers, $k \geqq u$, and $A \subset E_n$, then*

$$\int^*_{E_u \times G_n} \mathcal{H}_n^{k-u}[A \cap \lambda_n^{n-u}(R, z)] d(\mathcal{L}_u \otimes \phi_n)(z, R) \leqq \frac{\alpha(u)\alpha(k-u)}{\alpha(k)} \mathcal{H}_n^k(A).$$

**Proof.** We apply the preceding theorem with

$$P = A \times G_n,$$

and note that

$$(\mathcal{H}_n^k \otimes \phi_n)(P) = \mathcal{H}_n^k(A)\phi_n(G_n) = \mathcal{H}_n^k(A).$$

3.5 THEOREM. *If $v \geqq 0$ and*

$$P \subset E_n \times E_n \times G_n = E_{2n} \times G_n,$$

*then*

$$\int^*_{E_n \times G_n} \mathcal{H}_n^v(\{a \mid (a, (\operatorname{inv} R)(a - z), R) \in P\}) d(\mathcal{L}_n \otimes \phi_n)(z, R)$$

$$\leqq 2^{n/2} \frac{\alpha(n)\alpha(v)}{\alpha(n+v)} (\mathcal{H}_{2n}^{n+v} \otimes \phi_n)(P).$$

**Proof.** Let $f$ be the function on $E_n \times E_n \times G_n$ to $E_n$ such that

$$f(a, b, R) = a - R(b) \text{ for } (a, b, R) \in E_n \times E_n \times G_n.$$

Clearly $f$ is a continuous function and

$$|f(a, b, R) - f(a', b', R)| = |a - a' - R(b - b')|$$
$$\leqq |a - a'| + |b - b'| \leqq 2^{1/2} |(a, b) - (a', b')|$$

whenever $a$, $b$, $a'$, $b'$ are points of $E_n$ and $R \in G_n$.

Applying Theorem 3.2 with $X = E_{2n}$, $Y = G_n$, $Z = E_n$, $M = 2^{1/2}$, $u = n$, and

$\psi=\phi_n$ we obtain the inequality

$$\int^* \mathcal{H}_{2n}^v(\{(a, b) \mid (a, b, R) \in P \text{ and } a - R(b) = z\})d(\mathcal{L}_n \otimes \phi_n)(z, R)$$

$$\leqq 2^{n/2} \frac{\alpha(u)\alpha(v)}{\alpha(u+v)} (\mathcal{H}_{2n}^{n+v} \otimes \phi_n)(P).$$

Finally we consider a fixed point $(z, R) \in E_n \times G_n$ and note that

$$\begin{aligned}\{(a, b) \mid (a, b, R) \in P \text{ and } a - R(b) = z\} \\ &= \{(a, b) \mid (a, b, R) \in P \text{ and } b = (\text{inv } R)(a - z)\} \\ &= \{(a, (\text{inv } R)(a - z)) \mid (a, (\text{inv } R)(a - z), R) \in P\};\end{aligned}$$

hence this subset of $E_{2n}$ is mapped by the projection $p_n^{2n}$ onto the subset

$$\{a \mid (a, (\text{inv } R)(a - z), R) \in P\}$$

of $E_n$ and consequently

$$\begin{aligned}\mathcal{H}_n^v(\{a \mid (a, (\text{inv } R)(a - z), R) \in P\}) \\ &\leqq \mathcal{H}_{2n}^v(\{(a, b) \mid (a, b, R) \in P \text{ and } a - R(b) = z\}).\end{aligned}$$

3.6 COROLLARY. *If $A \subset E_n$, $B \subset E_n$, $k \geqq 0$, $l \geqq 0$, and $k+l \geqq n$, then*

$$\int^*_{E_n \times G_n} \mathcal{H}_n^{k+l-n}[A \cap (T_z \circ R)(B)]d(\mathcal{L}_n \otimes \phi_n)(z, R)$$

$$\leqq 2^{n/2} \frac{\alpha(n)\alpha(k+l-n)}{\alpha(k+l)} \mathcal{H}_{2n}^{k+l}(A \times B).$$

**Proof.** We apply Theorem 4.3 with $v=k+l-n$ and

$$P = A \times B \times G_n.$$

Since $\phi_n(G_n)=1$ we have

$$(\mathcal{H}_{2n}^{k+l} \otimes \phi_n)(P) = \mathcal{H}_{2n}^{k+l}(A \times B).$$

Furthermore if $(z, R) \in E_n \times G_n$, then

$$\begin{aligned}\{a \mid (a, (\text{inv } R)(a - z), R) \in P\} &= A \cap \{a \mid (\text{inv } R)(a - z) \in B\} \\ &= A \cap \{a \mid a - z \in R(B)\} \\ &= A \cap (T_z \circ R)(B).\end{aligned}$$

4. **On certain cartesian products.** It is known (see [BM]) that the $k+l$ dimensional Hausdorff measure of a cartesian product $A \times B$ can be infinite even though both the $k$ dimensional measure of $A$ and the $l$ dimensional measure of $B$ are finite. Here it will be shown that this pathological

situation never occurs in case $B$ is $l$ rectifiable. It will also be proved that if $A$ has $k$ dimensional measure zero and $B$ is $l$ rectifiable, then $A\times B$ has $k+l$ dimensional measure zero.

4.1 Lemma. *If $k>0$, $A\subset E_n$, and $B\subset E_l$, then*

$$\mathcal{H}^{k+l}_{n+l}(A \times B) \leq (l+1)^{(k+l)/2} 2^{-l} \frac{\alpha(k+l)}{\alpha(k)} \mathcal{H}^{k}_{n}(A)\mathcal{L}_l(B).$$

**Proof.** It is easy to reduce the general statement to the special case in which $B$ is an $l$ dimensional cube. We henceforth assume that $B$ is a cube with side length $b$. Letting

$$c = (l+1)^{(k+l)/2} 2^{-l} \frac{\alpha(k+l)}{\alpha(k)},$$

we divide the argument into two parts.

*Part* 1. *If $S\subset E_n$, then $S\times B$ has a finite covering $H$ such that*

$$\operatorname{diam}(T) = (l+1)^{1/2} \operatorname{diam}(S) \text{ for } T\in H$$

*and*

$$\sum_{T\in H} \chi^{k+l}_{n+l}(T) \leq c\chi^{k}_{n}(S)\,[b + \operatorname{diam}(S)]^l.$$

**Proof.** Let $r=\operatorname{diam}(S)$ and let $m$ be the least integer such that $rm\geq b$. Then

$$r(m-1) < b, \qquad rm < b + r,$$

and $B$ can be covered by $m^l$ cubes with side length $r$. Forming the cartesian product of $S$ with each of these cubes, we obtain a covering $H$ of $S\times B$ such that $H$ has $m^l$ elements and the diameter of each element of $H$ is equal to

$$(l+1)^{1/2} r.$$

It follows that

$$\begin{aligned}\sum_{T\in H} \chi^{k+l}_{n+l}(T) &= m^l \alpha(k+l) 2^{-(k+l)} (l+1)^{k+l/2} r^{k+l} \\ &= c\alpha(k) 2^{-k} r^k (rm)^l \leq c\chi^{k}_{n}(S)(b+r)^l.\end{aligned}$$

*Part* 2. *If $\epsilon>0$ and if $F$ is a countable covering of $A$ such that*

$$\sup_{S\in F} \operatorname{diam}(S) \leq \epsilon,$$

*then $A\times B$ has a countable covering $G$ such that*

$$\sup_{T\in G} \operatorname{diam}(T) \leq (l+1)^{1/2}\epsilon$$

*and*

$$\sum_{T\in G} \chi_{n+l}^{k+l}(T) \leqq c \sum_{S\in F} \chi_n^k(S)(b+\epsilon)^l.$$

**Proof.** Using Part 1 we associate with each $S\in F$ a covering $H_S$ of $S\times B$ such that

$$\text{diam } (T) \leqq (l+1)^{1/2}\epsilon \text{ for } T \in H_S$$

and

$$\sum_{T\in H_S} \chi_{n+l}^{k+l}(T) \leqq c\chi_n^k(S)(b+\epsilon)^l.$$

Then the family

$$G = \bigcup_{S\in F} H_S$$

is a suitable covering of $A\times B$.

Noting that $\mathcal{L}_l(B) = b^l$, we see that the inequality

$$\mathcal{H}_{n+l}^{k+l}(A\times B) \leqq c\,\mathcal{H}_n^k(A)\mathcal{L}_l(B)$$

is an immediate consequence of Part 2.

4.2 Theorem. *If $A\subset E_n$, $k>0$, and $B$ is an $l$ rectifiable subset of $E_p$, then*

(1) $\mathcal{H}_n^k(A)<\infty$ *implies that* $\mathcal{H}_{n+p}^{k+l}(A\times B)<\infty$,

(2) $\mathcal{H}_n^k(A)=0$ *implies that* $\mathcal{H}_{n+p}^{k+l}(A\times B)=0$.

**Proof.** Choosing a bounded set $C\subset E_l$ and a Lipschitzian function $f$ which maps $C$ onto $B$, we consider the function $g$ on $A\times C$ defined by the formula

$$g(x, y) = (x, f(y)) \text{ for } (x, y) \in A\times C.$$

Obviously the range of $g$ is $A\times B$. Furthermore if $M$ is a Lipschitz constant for $f$, then $M$ is also a Lipschitz constant for $g$ and consequently

$$\begin{aligned}\mathcal{H}_{n+p}^{k+l}(A\times B) &\leqq M^{k+l}\mathcal{H}_{n+l}^{k+l}(A\times C)\\ &\leqq M^{k+l}(l+1)^{(k+l)/2}2^{-l}\frac{\alpha(k+l)}{\alpha(k)}\mathcal{H}_n^k(A)\mathcal{L}_l(C).\end{aligned}$$

From this inequality the statements (1) and (2) follow immediately.

5. **The rectifiability of intersections.** In this section the connection between rectifiability and restrictedness is used to prove that a Hausdorff $k$ rectifiable subset of $E_n$ has Hausdorff $k-u$ rectifiable intersections with almost all $n-u$ dimensional planes, and also that a Hausdorff $k$ rectifiable

subset of $E_n$ has Hausdorff $k+l-n$ rectifiable intersections with almost all isometric images of an $l$ rectifiable set.

5.1 LEMMA. *If $X\subset E_n$, $Y\subset E_n$, $\theta$ is the origin of $E_n$, $U$ and $V$ are vector subspaces of $E_n$, $X$ is restricted at $\theta$ by $U$, and $Y$ is restricted at $\theta$ by $V$, then $X\cap Y$ is restricted at $\theta$ by $U\cap V$.*

**Proof.** Let $\epsilon>0$.

The non-negative continuous function $d_U+d_V$ vanishes only on $U\cap V$, and consequently has a positive lower bound $\lambda$ on the compact set

$$C = E_n \cap \{x \mid |x| = 1 \text{ and } d_{U\cap V}(x) \geqq \epsilon\}.$$

Let $0<\mu<\lambda/2$. If $\theta\neq x\in E_n$, then the condition

$$d_U(x) \leqq \mu |x| \quad \text{and} \quad d_V(x) \leqq \mu |x|$$

implies that

$$d_U\left(\frac{x}{|x|}\right) + d_V\left(\frac{x}{|x|}\right) < \lambda,$$

$$\frac{x}{|x|} \notin C, \qquad d_{U\cap V}\left(\frac{x}{|x|}\right) < \epsilon,$$

$$d_{U\cap V}(x) < \epsilon |x|.$$

Choosing a positive number $\delta$ for which

$$X \cap K(\theta, \delta) \subset \{x \mid d_U(x) \leqq \mu |x|\}$$

and

$$Y \cap K(\theta, \delta) \subset \{x \mid d_V(x) \leqq \mu |x|\},$$

we conclude that

$$X \cap Y \cap K(\theta, \delta) \subset \{x \mid d_U(x) \leqq \mu |x| \text{ and } d_V(x) \leqq \mu |x|\}$$
$$\subset \{x \mid d_{U\cap V}(x) \leqq \epsilon |x|\}.$$

5.2 LEMMA. *If $U$ and $V$ are vector subspaces of $E_n$ and*

$$\dim (U) + \dim (V) \geqq n,$$

*then*

$$\dim [U \cap R(V)] = \dim (U) + \dim (V) - n$$

*for $\phi_n$ almost all $R$ in $G_n$.*

**Proof.** We note that the set

$$G_n \cap \{R \mid \dim [U \cap R(V)] \neq \dim (U) + \dim (V) - n\}$$

contains neither component of $G_n$, and is equal to the set of all common zeros of certain determinants which are analytic functions on the Lie group $G_n$. (See [F4, p. 540].)

5.3 LEMMA. *If $A$ is a compact subset of $E_n$, $B$ is a closed subset of $E_n$, $m$ is a positive integer less than $n$, and $C$ is the set of all those triples*

$$(a, z, R) \in E_n \times E_n \times G_n$$

*for which the set*

$$A \cap (T_z \circ R)(B)$$

*is restricted at $a$ by some $m$ dimensional plane, then $C$ is an analytic set.*

**Proof.** For any positive integers $i, j, k$ we let

$$P_{i,j,k}$$

be the set of all those quadruples

$$(a, z, R, U) \in E_n \times E_n \times G_n \times \Lambda_n^m$$

for which

$$A \cap (T_z \circ R)(B) \cap \{x \mid k^{-1} \leqq |x - a| \leqq j^{-1}\} \subset \{x \mid d_U(x) < i^{-1}|x - a|\}.$$

We also define

$$S = (E_n \times E_n \times G_n \times \Lambda_n^m) \cap \{(a, z, R, U) \mid a \in U\}$$

and observe that $S$ is a closed set. Inasmuch as

$$S \cap \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} P_{i,j,k}$$

is equal to the set of all quadruples $(a, z, R, U)$ such that

$$A \cap (T_z \circ R)(B) \text{ is restricted at } a \text{ by } U,$$

it will be sufficient to show that each set $P_{i,j,k}$ is open.

For this purpose we note that the elements of the set

$$Q_{i,j,k} = (E_n \times E_n \times G_n \times \Lambda_n^m) - P_{i,j,k}$$

are those quadruples $(a, z, R, U)$ for which there exists a point $x$ satisfying the four conditions

$$x \in A, \quad x \in (T_z \circ R)(B), \quad k^{-1} \leqq |x - a| \leqq j^{-1}, \quad d_U(x) \geqq i^{-1}|x - a|.$$

Thus $Q_{i,j,k}$ is the projection image of a closed subset of the cartesian product space

$$E_n \times E_n \times G_n \times \Lambda_n^m \times A.$$

Since $A$ is compact it follows that $Q_{i,j,k}$ is closed and consequently that $P_{i,j,k}$ is open.

5.4 LEMMA. *If $A \subset E_n$, $a \in E_n$, $A$ is restricted at $a$ by some $k$ dimensional plane, and $u$ is a positive integer less than $k$, then for $\phi_n$ almost all $R$ in $G_n$ the set*

$$A \cap \lambda_n^{n-u}[R, (p_n^u \circ \text{inv } R)(a)]$$

*is restricted at $a$ by some $k-u$ dimensional plane.*

**Proof.** Let $X = T_{-a}(A)$. Since restrictedness is invariant under translations, the set $X$ is restricted at the origin $\theta$ of $E_n$ by a $k$ dimensional vector subspace $U$ of $E_n$. For each $R \in G_n$ we infer from Lemma 5.1, applied with $Y = V = R(Z_n^{n-u})$, that

$$X \cap R(Z_n^{n-u}) \text{ is restricted at } \theta \text{ by } U \cap R(Z_n^{n-u}).$$

Furthermore we know from Lemma 5.2 that for $\phi_n$ almost all $R$ the equations

$$\dim [U \cap R(Z_n^{n-u})] = k + (n - u) - n = k - u$$

is valid, so that $X \cap R(Z_n^{n-u})$ is restricted at $\theta$ by some $k-u$ dimensional plane. Finally we observe (see [F3, p. 539]) that if $R \in G_n$ then

$$T_a[X \cap R(Z_n^{n-u})] = A \cap \lambda_n^{n-u}[R, (p_n^u \circ \text{inv } R)(a)].$$

5.5 THEOREM. *If $A$ is a Hausdorff $k$ rectifiable subset of $E_n$ and $u$ is a positive integer less than $k$, then*

$$A \cap \lambda_n^{n-u}(R, z) \textit{ is Hausdorff } k - u \textit{ rectifiable}$$

*for $\mathcal{L}_u \otimes \phi_n$ almost all $(z, R)$ in $E_u \times G_n$.*

**Proof.** First we use Corollary 3.4 to infer that

$$\mathcal{H}_n^{k-u}[A \cap \lambda_n^{n-u}(R, z)] < \infty$$

for $\mathcal{L}_u \otimes \phi_n$ almost all $(z, R)$.

Next we choose $A_1, A_2, A_3, \cdots$ according to the proposition (2) of Remark 2.17 and let

$$A_0 = A - \bigcup_{i=1}^{\infty} A_i.$$

Applying Corollary 3.4 once more, we find that

$$\mathcal{H}_n^{k-u}[A_0 \cap \lambda_n^{n-u}(R, z)] = 0$$

for $\mathcal{L}_u \otimes \phi_n$ almost all $(z, R)$. To complete the proof we need therefore only show that if $i$ is a positive integer then

$$A \cap A_i \cap \lambda_n^{n-u}(R, z) \text{ is Hausdorff } k - u \text{ rectifiable}$$

for $\mathcal{L}_u \otimes \phi_n$ almost all $(z, R)$.

We henceforth consider a fixed positive integer $i$ and let $Q$ be the set of all those ordered pairs

$$(a, R) \in A_i \times G_n$$

for which the set

$$A_i \cap \lambda_n^{n-u}[R, (p_n^u \circ \text{inv } R)(a)] = A_i \cap (T_a \circ R)(Z_n^{n-u})$$

is restricted at $a$ by some $k-u$ dimensional plane.

Since Lemma 5.3 implies that $Q$ is an analytic subset of $A_i \times G_n$, we can use the Fubini Theorem in conjunction with Lemma 5.4 to obtain

$$(\mathcal{H}_n^k \otimes \phi_n)[(A_i \times G_n) - Q] = 0,$$

whence it follows by Theorem 3.3 that

$$\mathcal{H}_n^{k-u}[\{a \mid (a, R) \in (A_i \times G_n) - Q\} \cap \lambda_n^{n-u}(R, z)] = 0$$

for $\mathcal{L}_u \otimes \phi_n$ almost all $(z, R)$.

Now suppose $(z, R)$ is a point of $E_u \times G_n$ for which both

$$\mathcal{H}_n^{k-u}[A \cap \lambda_n^{n-u}(R, z)] < \infty$$

and

$$\mathcal{H}_n^{k-u}[\{a \mid (a, R) \in (A_i \times G_n) - Q\} \cap \lambda_n^{n-u}(R, z)] = 0,$$

and let

$$S = A \cap A_i \cap \lambda_n^{n-u}(R, z).$$

Then it is true for $\mathcal{H}_n^{k-u}$ almost all $a$ in $S$ that $(a, R) \in Q$; inasmuch as

$$(p_n^u \circ \text{inv } R)(a) = z,$$

it follows from the definition of $Q$ that $S$ is restricted at $a$ by some $k-u$ dimensional plane. Applying the proposition (1) of Remark 2.17 with $A$ and $k$ replaced by $S$ and $k-u$ respectively, we conclude that the set $S$ is Hausdorff $k-u$ rectifiable.

5.6 LEMMA. *If $A \subset E_n$, $B \subset E_n$, $a \in E_n$, $b \in E_n$, $A$ is restricted at $a$ by some $k$ dimensional plane, $B$ is restricted at $b$ by some $l$ dimensional plane, and $k+l>n$, then for $\phi_n$ almost all $R$ in $G_n$ the set*

$$A \cap [T_{a-R(b)} \circ R](B)$$

*is restricted at $a$ by some $k+l-n$ dimensional plane.*

**Proof.** Letting

$$X = T_{-a}(A), \qquad Y = T_{-b}(B),$$

we see that $X$ is restricted at the origin $\theta$ of $E_n$ by a $k$ dimensional vector subspace $U$ of $E_n$ and that $Y$ is restricted at $\theta$ by an $l$ dimensional vector subspace $V$ of $E_n$.

If $R \in G_n$, then $R(Y)$ is restricted at $\theta$ by $R(V)$ and, according to Lemma 5.1, $X \cap R(Y)$ is restricted at $\theta$ by $U \cap R(V)$; inasmuch as

$$T_{a-R(b)} \circ R = T_a \circ R \circ T_{-b},$$

the set

$$A \cap [T_{a-R(b)} \circ R](B) = T_a[X \cap R(Y)]$$

is restricted at $a$ by $T_a[U \cap R(V)]$.

Reference to Lemma 5.2 completes the proof.

5.7 THEOREM. *If $A$ is a Hausdorff $k$ rectifiable subset of $E_n$, $B$ is an $l$ rectifiable subset of $E_n$, and $k+l>n$, then*

$$A \cap (T_z \circ R)(B) \text{ is Hausdorff } k + l - n \text{ rectifiable}$$

*for $\mathcal{L}_n \otimes \phi_n$ almost all $(z, R)$ in $E_n \times G_n$.*

**Proof.** First we use the statement (1) of Theorem 4.2 to obtain the inequality

$$\mathcal{H}_{2n}^{k+l}(A \times B) < \infty,$$

and infer from Corollary 3.6 that

$$\mathcal{H}_n^{k+l-n}[A \cap (T_z \circ R)(B)] < \infty$$

for $\mathcal{L}_n \otimes \phi_n$ almost all $(z, R)$.

Next we choose $A_1, A_2, A_3, \cdots$ according to the proposition (2) of Remark 2.17, and similarly select compact $l$ rectifiable subsets $B_1, B_2, B_3, \cdots$ of $E_n$ such that

$$\mathcal{H}_n^l\left(B - \bigcup_{j=1}^{\infty} B_j\right) = 0$$

and such that for each positive integer $j$ and each point $b \in B_j$ it is true that $B_j$ is restricted at $b$ by some $l$ dimensional plane. Letting

$$A_0 = A - \bigcup_{i=1}^{\infty} A_i, \qquad B_0 = B - \bigcup_{j=1}^{\infty} B_j,$$

we apply the statement (2) of Theorem 4.2 to infer that

$$\mathcal{H}_{2n}^{k+l}(A_0 \times B) = 0$$

and

$$\mathcal{H}_{2n}^{k+l}(A_i \times B_0) = \mathcal{H}_{2n}^{l+k}(B_0 \times A_i) = 0 \text{ for } i = 1, 2, 3, \cdots .$$

Accordingly Corollary 3.6 implies that the conditions

$$\mathcal{H}_n^{k+l-n}[A_0 \cap (T_z \circ R)(B)] = 0$$

and

$$\mathcal{H}_n^{k+l-n}[A_i \cap (T_z \circ R)(B_0)] = 0 \quad \text{for } i = 1, 2, 3, \cdots$$

hold for $\mathcal{L}_n \otimes \phi_n$ almost all $(z, R)$. Inasmuch as

$$[A \cap (T_z \circ R)(B)] \subset [A_0 \cap (T_z \circ R)(B)] \cup \bigcup_{i=1}^{\infty} [A_i \cap (T_z \circ R)(B_0)]$$
$$\cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} [A \cap A_i \cap (T_z \circ R)(B_j)]$$

whenever $(z, R) \in E_n \times G_n$, the proof will be complete as soon as we have shown that if $i$ and $j$ are positive integers then

$$A \cap A_i \cap (T_z \circ R)(B_j) \text{ is Hausdorff } k + l - n \text{ rectifiable}$$

for $\mathcal{L}_n \otimes \phi_n$ almost all $(z, R)$.

We henceforth consider fixed positive integers $i$ and $j$ and let $Q$ be the set of all those triples

$$(a, b, R) \in A_i \times B_j \times G_n$$

for which the set

$$A_i \cap [T_{a-R(b)} \circ R](B_j)$$

is restricted at $a$ by some $k+l-n$ dimensional plane.

Since Lemma 5.3 implies that $Q$ is an analytic subset of the cartesian product space

$$A_i \times B_j \times G_n = (A_i \times B_j) \times G_n,$$

and since

$$\mathcal{H}_{2n}^{k+l}(A_i \times B_j) < \infty,$$

according to the statement (1) of Theorem 4.2, we can use the Fubini Theorem in conjunction with Lemma 5.6 to obtain

$$(\mathcal{H}_{2n}^{k+l} \otimes \phi_n)[(A_i \times B_j \times G_n) - Q] = 0.$$

Applying Theorem 3.5 with $v=k+l-n$, we find that

$$\mathcal{H}_n^{k+l-n}[\{a \mid (a, (\text{inv } R)(a-z), R) \in (A_i \times B_j \times G_n) - Q] = 0$$

for $\mathcal{L}_n \otimes \phi_n$ almost all $(z, R)$.

Now suppose $(z, R)$ is a point of $E_n \times G_n$ for which both

$$\mathcal{H}_n^{k+l-n}[A \cap (T_z \circ R)(B)] < \infty$$

and

$$\mathcal{H}_n^{k+l-n}[\{a \mid (a, (\text{inv } R)(a-z), R) \in (A_i \times B_j \times G_n) - Q\}] = 0,$$

and let

$$S = A \cap A_i \cap (T_z \circ R)(B_j).$$

We observe that if $a \in S$ then

$$a \in (T_z \circ R)(B_j), \qquad a - z \in R(B_j), \qquad (\text{inv } R)(a-z) \in B_j,$$
$$(a, (\text{inv } R)(a-z), R) \in A_i \times B_j \times G_n.$$

Therefore it is true for $\mathcal{H}_n^{k+l-n}$ almost all $a$ in $S$ that

$$(a, (\text{inv } R)(a-z), R) \in Q;$$

inasmuch as

$$a - R[(\text{inv } R)(a-z)] = z,$$

it follows from the definition of $Q$ that $S$ is restricted at $a$ by some $k+l-n$ dimensional plane. Applying the proposition (1) of Remark 2.17 with $A$ and $k$ replaced by $S$ and $k+l-n$ respectively, we conclude that the set $S$ is Hausdorff $k+l-n$ rectifiable.

6. **The principal integralgeometric formula.** Considering again the intersections of a Hausdorff $k$ rectifiable subset of $E_n$ with the isometric images of an $l$ rectifiable set, we compute here the integral over the group of isometries of the $k+l-n$ dimensional Hausdorff measures of these intersections.

6.1 LEMMA. *If $A$ and $B$ are analytic subsets of $E_n$, $A$ is Hausdorff $k$ rectifiable, and $B$ is $n-k$ rectifiable, then*

$$\int_{E_n \times G_n} \mathcal{H}_n^0[A \cap (T_z \circ R)(B)]\, d(\mathcal{L}_n \otimes \phi_n)(z, R) = \beta(n, k)\, \mathcal{H}_n^k(A)\, \mathcal{H}_n^{n-k}(B).$$

**Proof.** First we note, as a consequence of Theorem 4.1 of [F1], that the integrand in the above formula is an $\mathcal{L}_n \otimes \phi_n$ measurable function. In fact if $f$ is the function on $A \otimes B \otimes G_n$ to $E_n \otimes G_n$ such that

$$f(a, b, S) = (a - S(b), S) \text{ for } (a, b, S) \in A \times B \times G_n,$$

and if $(z, R) \in E_n \times G_n$, then the multiplicity with which $f$ assumes the value $(z, R)$ is equal to the number of elements of the set

$$A \cap (T_z \circ R)(B).$$

Inasmuch as $\mathcal{H}_n^k$ almost all of $A$ is the union of countably many disjoint analytic $k$ rectifiable sets, it will obviously be sufficient to show that the formula is valid in case either $\mathcal{H}_n^k(A) = 0$ or $A$ is $k$ rectifiable.

If $\mathcal{H}_n^k(A) = 0$, then the formula is a consequence of Theorem 4.2 and Corollary 3.6.

If $A$ is $k$ rectifiable, then $A$ and $B$ can be represented as univalent (see [MF]) Lipschitzian images of an $\mathcal{L}_k$ measurable subset of $E_k$ and an $\mathcal{L}_{n-k}$ measurable subset of $E_{n-k}$ respectively, and the formula follows from Theorem 4.2 of [F2] and Theorem 5.9 of [F3].

6.2 THEOREM. *If $A$ and $B$ are analytic subsets of $E_n$, $A$ is Hausdorff $k$ rectifiable, $B$ is $l$ rectifiable, and $k+l \geqq n$, then*

$$\int_{E_n \times G_n} \mathcal{H}_n^{k+l-n}[A \cap (T_z \circ R)(B)] d(\mathcal{L}_n \otimes \phi_n)(z, R) = \gamma(n, k, l)\, \mathcal{H}_n^k(A)\, \mathcal{H}_n^l(B).$$

**Proof.** In view of the preceding lemma we may assume that $k+l > n$. Letting $h$ be the function on $E_{k+l-n} \times G_n \times E_n \times G_n$ such that

$$h(y, S, z, R) = \mathcal{H}_n^0[A \cap \lambda_n^{2n-k-l}(S, y) \cap (T_z \circ R)(B)]$$

whenever $y \in E_{k+l-n}$, $S \in G_n$, $z \in E_n$, and $R \in G_n$, we divide our argument into three parts; the theorem is an immediate consequence of Part 2 and Part 3.

*Part* 1. *$h$ is an $\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n$ measurable function.*

**Proof.** Considering the function

$$f: A \times B \times G_n \times G_n \to E_{k+l-n} \times G_n \times E_n \times G_n$$

such that

$$f(a, b, U, V) = ((p_n^{k+l-n} \circ \operatorname{inv} U)(a), U, a - V(b), V)$$

whenever $a \in A$, $b \in B$, $U \in G_n$ and $V \in G_n$, we note that if

$$(y, S, z, R) \in E_{k+l-n} \times G_n \times E_n \times G_n,$$

then the multiplicity with which $f$ assumes the value $(y, S, z, R)$ is equal to $h(y, S, z, R)$. Hence Part 1 is a consequence of Theorem 4.1 of [F1].

*Part* 2.

$$\int \mathcal{H}_n^{k+l-n}[A \cap (T_z \circ R)(B)] d(\mathcal{L}_n \otimes \phi_n)(z, R)$$

$$= \frac{1}{\beta(n, k+l-n)} \int h\, d(\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n).$$

**Proof.** According to Part 1 and the Fubini Theorem we have

$$\int hd(\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n)$$

$$= \int\int h(y, S, z, R)d(\mathcal{L}_{k+l-n} \otimes \phi_n)(y, S)d(\mathcal{L}_n \otimes \phi_n)(z, R).$$

Furthermore Theorem 5.7 implies that for $\mathcal{L}_n \otimes \phi_n$ almost all $(z, R)$ the set

$$A \cap (T_z \circ R)(B)$$

is $k+l-n$ rectifiable, whence it follows by Theorem 9.7 of [F3] and by the formula for the integralgeometric measure given in Section 5 of [F4] that

$$\mathcal{H}_n^{k+l-n}[A \cap (T_z \circ R)(B)] = \mathcal{I}_n^{k+l-n}[A \cap (T_z \circ R)(B)]$$

$$= \frac{1}{\beta(n, k + l - n)} \int h(y, S, z, R)d(\mathcal{L}_{k+l-n} \otimes \phi_n)(y, S).$$

*Part* 3.

$$\int hd(\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n) = \frac{\beta(n, k)\beta(n, l)}{\beta(2n - k - l, n - l)} \mathcal{H}_n^k(A)\mathcal{H}_n^l(B).$$

**Proof.** According to Part 1 and the Fubini Theorem we have

$$\int hd(\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n)$$

$$= \int\int h(y, S, z, R)d(\mathcal{L}_n \otimes \phi_n)(z, R)d(\mathcal{L}_{k+l-n} \otimes \phi_n)(y, S).$$

Furthermore Theorem 5.5 implies that for $\mathcal{L}_{k+l-n} \otimes \phi_n$ almost all $(y, S)$ the set

$$A \cap \lambda_n^{2n-k-l}(S, y)$$

is $n-l$ rectifiable, whence it follows by Lemma 6.1 that

$$\int h(y, S, z, R)d(\mathcal{L}_n \otimes \phi_n)(z, R) = \beta(n, l)\mathcal{H}_n^{n-l}[A \cap \lambda_n^{2n-k-l}(S, y)]\mathcal{H}_n^l(B)$$

$$= \beta(n, l)\mathcal{I}_n^{n-l}[A \cap \lambda_n^{2n-k-l}(S, y)]\mathcal{H}_n^l(B).$$

Finally we use the theorem in Section 8 of [F4] with $m = n - l$ to compute

$$\int \mathcal{I}_n^{n-l}[A \cap \lambda_n^{2n-k-l}(S, y)]d(\mathcal{L}_{k+l-n} \otimes \phi_n)(y, S)$$

$$= \frac{\beta(n, k)}{\beta(2n - k - l, n - l)} \mathcal{I}_n^k(A) = \frac{\beta(n, k)}{\beta(2n - k - l, n - l)} \mathcal{H}_n^k(A).$$

6.3 REMARK. The condition "$A$ and $B$ are analytic subsets of $E_n$" in Theorem 6.2 can be replaced by the weaker condition "$A$ is $\mathcal{H}_n^k$ measurable and $B$ is $\mathcal{H}_n^l$ measurable." This extension presents no difficulty because every Hausdorff measurable set of finite measure contains an $F_\sigma$ of equal measure, and is contained in a $G_\delta$ of equal measure.

The problem whether or not the condition "$B$ is $l$ rectifiable" can be replaced by the weaker condition "$B$ is Hausdorff $l$ rectifiable" is still unsolved, even for the simplest case in which $k=1$, $l=1$ and $n=2$.

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